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On the classical solution for the double-brane background in open string field theory

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Abstract

Murata and Schnabl recently proposed solutions for the multiple-brane backgrounds in Witten's open bosonic string field theory. The solutions contain some singularities, and in one particular regularization, the double-brane solution reproduces the desired energy and the Ellwood invariant, which is conjectured to represent coupling to a closed string. However, it turned out that the equation of motion is slightly violated. In this paper, we propose another regularization method for the double-brane solution. The regularized solution is realized as a superposition of the wedge states with operator insertions. It respects the equation of motion both contracted with the solution itself and with the states in the Fock space. It reproduces the desired double-brane tension, while the expected Ellwood invariant is not obtained.

Contents

1 Introduction

Schnabl's construction [1] of the analytic solution for tachyon condensation provided us with a set of new techniques to analyze Witten's bosonic, open, cubic string field theory (CSFT) [3]. In [4], Okawa rewrote the original derivation and abstracted an important concept called KBc subalgebra (see also [5]). It consists of three string fields, K, B and c, satisfying[1]

$$[K, B] = 0\,, \qquad \{B, c\} = 1\,, \qquad B^2 = 0\,, \qquad c^2 = 0\,, \tag{1.1}$$

and

$$dB = K\,, \qquad dK = 0\,, \qquad dc = cKc\,, \tag{1.2}$$

where d is a derivation operator representing BRST transformation. Algebraic relations (1.1) and (1.2) look quite simple, especially compared with the general Neumann or CFT method for the star product calculations. However, they are sufficient to write down the tachyon vacuum and pure-gauge

[1] The $*$-symbol for the star product is implicit in this paper.

solutions [6,7]. Furthermore, with slight generalizations, we can also write down the marginal solutions with them [8–11].

In spite of the recent progress of the theory of the analytic calculation, various important things remain more or less obscure, notably, the existence of the multiple-brane solutions. The main concern of the present paper is the multiple-brane solutions presented in [12, 13] by Murata and Schnabl, especially the double-brane solution [2]

$$\Psi = \frac{1}{K} c \frac{KB}{K-1} c \,. \tag{1.3}$$

It is easy to check that Ψ in (1.3) is a solution *in a formal sense* to the string field equation of motion (eom)

$$d\Psi + \Psi\Psi = 0 \,. \tag{1.4}$$

The problem is that whether Ψ is realized as a *true, physical* solution or not. We have several reasons to believe that the solution (1.3) represents the double-brane background, as explained in [12, 13] by Murata and Schnabl, and in [14] by Hata and Kojita. These proceeding works are full of suggestions, but we are still far from the final settlement since the expression (1.3) contains a singular factor, $1/K$, as explained later. Hence, we need to find a suitable regularization method.

Formal solutions to the KBc subalgebra

When speaking about KBc, it is convenient to distinguish between two frameworks of discussion; one is based on a formal structure of KBc subalgebra, and the other is based on true, physical string fields. The former is a purely algebraic framework. We only use the algebraic relations, (1.1) and (1.2), and do not care about the content of each element. In particular, we admit an arbitrary function of K, $f(K)$, and its inverse, $f^{-1}(K)$. We can discuss the eom or formal gauge transformations within it, but cannot discuss the energy since the inner product is not defined. The following is a formal solution to the eom for an arbitrary function $F(K)$:

$$\Psi_{Okawa} = F(K)^2 c \frac{K}{1-F(K)^2} Bc \,. \tag{1.5}$$

Note that if we set $F(K)^2 = 1/K$, we obtain the solution (1.3).

On the other hand, in the true story, K, B and c have definite meanings as string fields. The inner product (2.5) and (2.6) is defined for wedge-based states [15] , which is a special case of the general

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[2] We basically follow the notation of Okawa [4], which is different from that used in [12, 13] by Murata and Schnabl. See Section 2 of [4]. However, our K and B are $\pi/2$ times those used in [4], while our c is $2/\pi$ times that in [4].
When we care about the reality condition, we should write Ψ as

$$\Psi = \sqrt{\frac{1}{-K}} c \frac{KB}{K-1} c \sqrt{\frac{1}{-K}} \,,$$

which is connected to (1.3) by a gauge transformation.

inner product of the string fields. The state space of the physical KBc subalgebra is unclear, just as the state space of the whole string fields is so. However, once we employ the definition of inner product (2.5) and (2.6), solutions to the eom are subjected to a strict constraint; namely, *a physical solution to the eom should produce a definite energy density based on (2.5) and (2.6).*

The energy formula by Murata and Schnabl

Using some analytic continuation argument and some algebraic identities, such as

$$(1 - F(K)^2) * \frac{1}{1 - F(K)^2} = 1\,, \tag{1.6}$$

Murata and Schnabl calculated the energy $E(\Psi)$ of the solution (1.5) and derived the following formula [12, 13]:

$$E(\Psi_{Okawa}) = -\lim_{x \to 0} \frac{x(1 - F(x)^2)'}{1 - F(x)^2}\,. \tag{1.7}$$

This formula reproduces correct results for known tachyon vacuum solutions and perturbative vacuum solutions. If the function $G(x) = 1 - F^2(x)$ has a pole of order n at $x = 0$, then the energy of the solution is given by

$$E = \frac{n}{2\pi^2}\,. \tag{1.8}$$

Therefore, the string field (1.3) is expected to be a double-brane solution.[3]

However, it is still unclear whether the solution (1.3) reproduces a definite energy value or not. That is, when we calculate the energy based on (2.5) and (2.6), we have to represent $1/K$ as a superposition of the wedge states. If we use the following expression

$$\frac{1}{K} \stackrel{?}{=} -\int_0^\infty dt\, e^{tK}\,, \tag{1.9}$$

then it is not clear whether the identity (1.6) holds or not. Indeed, without regularizations, energy of the solution becomes indefinite, although it always remain finite, because of the integration over the parameter t, as we will see in Section 3.

Regularization of the double-brane solution[4]

Let us consider a simple situation. If we regularize $1/K$ as

$$\lim_{\epsilon \to 0} \frac{1}{K - \epsilon} = -\lim_{\epsilon \to 0} \int_0^\infty dt e^{t(K-\epsilon)}\,, \tag{1.10}$$

[3] In our notation, the D-brane tension is normalized to be $1/(2\pi^2)$.

[4] In passing, let us here clarify the meaning of the word *regularization*. When we regularize the solution Ψ, we consider a one parameter family of regular string fields $\{\Psi_\Lambda\}$, and try to define a double-brane solution by the limit, $\lim_{\Lambda\to\infty} \Psi_\Lambda$. It is similar to the case of the Dirac delta function. In many applications, the delta function is regarded as a weak limit of a sequence of smooth functions. So it is natural to take the limit of the parameter Λ after all the other calculations.

then the regularized solution,

$$\Psi_{e^{-\epsilon t}} = \lim_{\epsilon\to 0} \frac{1}{K-\epsilon} c \frac{K^2}{K-1} Bc\,, \tag{1.11}$$

does not satisfy the equation of motion. A source of the problem might be the remainder of the algebraic relation,

$$\lim_{\epsilon\to 0} \frac{1}{K-\epsilon} * K = 1 - \lim_{\epsilon\to 0} \frac{-\epsilon}{K-\epsilon}\,. \tag{1.12}$$

The second term on the right hand side has some sliver-like property[5], as mentioned in [17] by Bonora et al. and further discussed in [18] by Erler and Maccaferri, and therefore, it is not zero. In fact, it does contribute to the calculation of the kinetic term and cubic term, and breaks the eom contracted with the solution itself.

The problem can be alleviated using a method called ϵ-trick (or, also called K_ϵ-regularization in [14],) given by Murata and Schnabl [12], and by Hata and Kojita [14]. The regularized double-brane solution becomes as follows:

$$\Psi_{\epsilon\text{-}trick} = \lim_{\epsilon\to 0} \frac{1}{K-\epsilon} c \frac{(K-\epsilon)^2}{(K-\epsilon)-1} Bc\,. \tag{1.13}$$

The solution $\Psi_{\epsilon\text{-}trick}$ satisfies the equation of motion contracted with the solution itself, and also produces the desired double-brane tension. However, it does not satisfy the eom contracted with the states in the Fock space, as explained in [12].

Summary of the present paper

In this paper we regularize the solution (1.3) to satisfy the eom. Let us briefly explain the key idea of our regularization. Suppose we want to regularize an integral whose value is finite but indefinite,

$$I = \int_0^\infty dx \int_0^\infty dy\, f(x,\, y)\,, \tag{1.14}$$

where $f(x,\, y)$ is a function with some appropriate properties. For a typical example, consider

$$f(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y} \sin \frac{\pi x}{2(x+y)}.$$

If we take the following convergence factor,

$$R_0(\Lambda;\, x) = \begin{cases} 1 - \ln(x+1)/\ln(\Lambda+1) & (0 \le x \le \Lambda) \\ 0 & (\Lambda < x)\,, \end{cases} \tag{1.15}$$

[5]The sliver state $e^{\infty K}$ is sometimes thought to be an 'eigenvector' of K with the eigenvalue zero. (The real things are somewhat more complicated because of the fact that some inner products including $Ke^{\infty K}$ is not zero (or, at least indefinite) depending on the order of limit operations.) If we consider K as a real number, the second term of (1.12) vanishes for $K < 0$ and becomes nonvanishing for $K = 0$.

then the integral is equivalent to that calculated under the symmetric-cutoff prescription:

$$\begin{aligned} I_{R_0} =& \lim_{\Lambda\to\infty}\int_0^\infty R_0(\Lambda;\, x)dx\int_0^\infty R_0(\Lambda;\, y)dy\, f(x,\, y) \\ =& \frac{1}{2}\left(\lim_{a\to 0}+\lim_{a\to\infty}\right)\lim_{\Lambda\to\infty}\int_0^\Lambda dx\int_0^{a\Lambda} dy\, f(x,\, y)\,. \end{aligned} \tag{1.16}$$

We will make this statement precise in Section 4.

Using this relation, we show that the solution

$$\Psi_{R_0} = \lim_{\Lambda\to\infty} K^{-1}_{R_0}(\Lambda)c\frac{K^2}{K-1}Bc\,, \tag{1.17}$$

with

$$K^{-1}_{R_0}(\Lambda) = -\int_0^\infty R_0(\Lambda;\, x)e^{xK}dx\,, \tag{1.18}$$

satisfies the eom contracted with Ψ_{R_0} itself, but it does not satisfy the eom contracted with some states in the Fock space. To describe the situation, it is convenient to use the $\mathcal{L}_0$ Fock basis. The eom contracted to the $\mathcal{L}_0$ Fock basis are all satisfied except for certain two states. To recover the eom, we need to add a correction term called a phantom piece φ_p given by

$$\varphi_p = r\int_0^\infty d\lambda\frac{dR_*(\Lambda;\, \lambda)}{d\lambda}\left(\frac{1}{K}\right)_\lambda cKe^{\lambda K}BcK\,, \tag{1.19}$$

where

$$R_*(\Lambda;\, \lambda) = R_1(\Lambda,\, \lambda) + \frac{i}{3^{\Lambda^2}}\, R_1(\Lambda,\, \tfrac{\lambda}{3^{2\Lambda^2}})\,, \tag{1.20}$$

$$R_1(\Lambda,\, x) = \begin{cases} 2-(x+1)^{\frac{1}{\Lambda^2}} & (0\le x\le 2^{\Lambda^2}-1)\,, \\ 0 & (2^{\Lambda^2}-1<x)\,, \end{cases} \tag{1.21}$$

$$r = -\frac{2-\gamma+\mathrm{Ci}(2\pi)-\ln(2\pi)}{2-4\mathrm{Ci}(\pi)+4\mathrm{Ci}(2\pi)-2\ln 4}\,. \tag{1.22}$$

The regularized solution

$$\dot{\Psi} = \Psi_{R_0}+\varphi_p$$

satisfies the equation of motion both contracted with the solution itself and with all the states in the Fock space. It also reproduces the double-brane tension, but it does not reproduce the desired Ellwood invariant. In fact, it is for the pure-gauge solutions.

This paper is organized as follows. In Section 2, we briefly review the KBc subalgebra and derive some useful formulas for correlation functions, such as (2.22) and (2.24). In Section 3, we consider a simple regularization of $1/K$ with a cutoff parameter Λ and evaluate the action. We will see that the regularized solution does not satisfy the eom. In Section 4, we regularize $1/K$ with general convergence factor $R(\Lambda;\, x)$. We evaluate the action using the results of Section 3 and find that for some class of the convergence factors, R_0 in (1.15) for example, the solution Ψ_{R_0} satisfies the eom contracted with

the solution itself and reproduces the desired double-brane tension. In Section 5, we investigate the eom in the Fock space. We see that Ψ_{R_0} does not satisfy the eom on the Fock space. In Section 5.3, we give an expression for the phantom piece, φ_p. The string field $\dot{\Psi} = \Psi_{R_0} + \varphi_p$ satisfies the eom both contracted with the solution itself and the states in the Fock space. Section 6 is devoted to the discussion on the component fields and Ellwood invariants. Section 7 is for concluding remarks.

2 KBc subalgebra

The purpose of this section is to briefly introduce the KBc subalgebra and to determine a notation. For detailed description of the KBc subalgebra, see [4, 5] for example.

Consider a class of string fields consist of K, B, c, where

$$K : \text{Grassmann even}\,, \qquad B : \text{Grassmann odd}\,, \qquad c : \text{Grassmann odd}\,. \tag{2.1}$$

They satisfy the following relations:

$$[\,K,\,B\,] = 0\,, \qquad \{\,B,\,c\,\} = 1\,, \qquad B^2 = 0\,, \qquad c^2 = 0\,. \tag{2.2}$$

The BRST transformation corresponds to a derivation d

$$dB = K\,, \qquad dK = 0\,, \qquad dc = cKc\,, \tag{2.3}$$

$$d\,(\Phi_1\Phi_2) = (d\Phi_1)\,\Phi_2 + (-1)^{\Phi_1}\Phi_1\,d\Phi_2\,. \tag{2.4}$$

Here and in what follows a string fields in the exponent of (-1) denotes its Grassmann property; it is 0 mod 2 for a Grassmann-even string field and it 1 mod 2 for a Grassmann-odd string field.

The trace corresponding to the correlation function is given by

$$\begin{aligned}\mathrm{tr}[ce^{xK}ce^{yK}ce^{zK}] &= -\left(\frac{x+y+z}{\pi}\right)^3 \sin\frac{\pi x}{x+y+z}\sin\frac{\pi y}{x+y+z}\sin\frac{\pi z}{x+y+z}\\ &= -\frac{1}{4}\left(\frac{x+y+z}{\pi}\right)^3\left(\sin\frac{2\pi x}{x+y+z}+\sin\frac{2\pi y}{x+y+z}+\sin\frac{2\pi z}{x+y+z}\right),\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}\mathrm{tr}[Bce^{x_1K}ce^{x_2K}ce^{x_3K}ce^{x_4K}] = -\frac{s^2}{4\pi^3}\Big(&x_3\sin\frac{2\pi x_1}{s} - (x_2+x_3)\sin\frac{2\pi(x_1+x_2)}{s} + x_2\sin\frac{2\pi(x_1+x_2+x_3)}{s}\\ &+ x_1\sin\frac{2\pi x_3}{s} - (x_1+x_2)\sin\frac{2\pi(x_2+x_3)}{s} + (x_1+x_2+x_3)\sin\frac{2\pi x_2}{s}\Big)\\ \equiv\, & F(x_1,\,x_2,\,x_3,\,x_4)\end{aligned} \tag{2.6}$$

with

$$s = x_1 + x_2 + x_3 + x_4. \tag{2.7}$$

Note that the trace of a BRST exact object is zero[6]

$$\operatorname{tr} d(\ldots) = 0\,. \tag{2.8}$$

The string field equation of motion is

$$d\Psi + \Psi^2 = 0\,. \tag{2.9}$$

Let $\hat{V}$ denote a normalized potential,

$$\widehat{V}(\Psi) = 2\pi^2 \operatorname{tr}\left[\frac{1}{2}\Psi d\Psi + \frac{1}{3}\Psi^3\right]. \tag{2.10}$$

The quantity $\widehat{V}+1$ represents the multiplicity of the branes. For example,

$$\widehat{V}(\Psi) = \begin{cases} 0 & \text{for pure gauge solutions,} \\ -1 & \text{for tachyon vacuum solutions.} \end{cases} \tag{2.11}$$

For later convenience, let us define the functions $\widehat{V}_K$, $\widehat{V}_C$ as follows:

$$\widehat{V}_K(\Psi_1,\, \Psi_2) = \frac{\pi^2}{3}\operatorname{tr}\left[\,\Psi_1 d\Psi_2\,\right], \tag{2.12}$$

$$\widehat{V}_C(\Psi_1,\, \Psi_2,\, \Psi_3) = -\frac{\pi^2}{3}\operatorname{tr}\left[\,\Psi_1\Psi_2\Psi_3\,\right]. \tag{2.13}$$

If Ψ is a solution to the eom, the following relation holds:

$$\widehat{V}(\Psi) = \widehat{V}_K(\Psi,\, \Psi) = \widehat{V}_C(\Psi,\, \Psi,\, \Psi)\,. \tag{2.14}$$

A class of formal solutions to the eom (2.9) is obtained by Okawa [4], which is a generalization of the tachyon vacuum solution by Schnabl [1],

$$\Psi = F(K)c\frac{KB}{1-F^2(K)}cF(K)\,, \tag{2.15}$$

where $F(K)$ is a function of K. When we do not care about the reality condition of the solution, we often use simpler non-real form,

$$\Psi = F(K)^2 c\frac{KB}{1-F^2(K)}c\,. \tag{2.16}$$

In order to calculate the energy of solutions, we suppose that the solution can be written as a superposition of the wedge states.

$$\begin{aligned} \Psi &= F(K)^2 c\frac{KB}{1-F(K)^2}c \\ &= \int_0^\infty dx \int_0^\infty du \tilde{F}(x)\tilde{H}(u)e^{xK}ce^{uK}Bc\,, \end{aligned} \tag{2.17}$$

[6]We assume that the string fields represented as $\ldots$ in (2.8) do not include any singular objects such as $1/K$.

where

$$F(K)^2 = \int_0^\infty dx \tilde{F}(x) e^{xK}\,, \tag{2.18}$$

$$H(K) = \frac{K}{1-F(K)^2} = \int_0^\infty du \tilde{H}(u) e^{uK}\,. \tag{2.19}$$

The potential for the solution Ψ is then written as follows:

$$\begin{aligned}\widehat{V}_K(\Psi,\,\Psi) =& \frac{\pi^2}{3}\int_0^\infty dx\int_0^\infty dy\int_0^\infty du\int_0^\infty dv \tilde{F}(x)\tilde{F}(y)\tilde{H}(u)\tilde{H}(v)\\ &\times\,\mathrm{tr}\,[\,e^{xK}ce^{uK}Bcd(e^{yK}ce^{vK}Bc)\,]\,,\end{aligned} \tag{2.20}$$

$$\begin{aligned}\widehat{V}_C(\Psi,\,\Psi,\,\Psi) =& \frac{\pi^2}{3}\int_0^\infty dx\int_0^\infty dy\int_0^\infty du\int_0^\infty dv \tilde{F}(x)\tilde{F}(y)\tilde{F}(z)\tilde{H}(u)\tilde{H}(v)\tilde{H}(w)\\ &\times\,\mathrm{tr}\,[\,e^{xK}ce^{uK}Bc\,e^{yK}ce^{vK}Bc\,e^{zK}ce^{wK}Bc\,]\,.\end{aligned} \tag{2.21}$$

The explicit form of the integration kernel $C_K(x,\,y;\,u,\,v)$ in the expression (2.20) is

$$\begin{aligned}&C_K(x,y;u,v) \equiv \mathrm{tr}[e^{xK}ce^{uK}Bcd(e^{yK}ce^{vK}Bc)]\\ &=\frac{1}{2\pi^2}\Bigg\{-(x+y)s + y(s-x)\cos\frac{2\pi x}{s} + x(s-y)\cos\frac{2\pi y}{s} + uv\cos\frac{2\pi u}{s} + uv\cos\frac{2\pi v}{s}\\ &+(xy-uv)\cos\frac{2\pi(x+v)}{s} + (xy-uv)\cos\frac{2\pi(y+v)}{s}\Bigg\}\\ &+\frac{s}{4\pi^3}\Bigg\{2y\sin\frac{2\pi x}{s} + 2x\sin\frac{2\pi y}{s} + (s-2v)\sin\frac{2\pi u}{s} + (s-2u)\sin\frac{2\pi v}{s}\\ &+(x-y+u-v)\sin\frac{2\pi(x+v)}{s} + (x-y-u+v)\sin\frac{2\pi(y+v)}{s}\Bigg\}\,,\end{aligned} \tag{2.22}$$

with

$$s = x+y+u+v\,. \tag{2.23}$$

The integration kernel for the cubic term (2.21) is

$$\begin{aligned}&C_C(x,y,z;u,v,w)\\ =&\mathrm{tr}[e^{xK}ce^{uK}Bce^{yK}cBe^{vK}ce^{zK}cBe^{wK}c]\\ =&\frac{s^2}{4\pi^3}x\left(\sin\frac{2\pi v}{s} - \sin\frac{2\pi(v+y)}{s} - \sin\frac{2\pi(v+z)}{s} + \sin\frac{2\pi(v+y+z)}{s}\right)\\ &+\frac{s^2}{4\pi^3}y\left(\sin\frac{2\pi w}{s} - \sin\frac{2\pi(w+z)}{s} - \sin\frac{2\pi(w+x)}{s} + \sin\frac{2\pi(w+z+x)}{s}\right)\\ &+\frac{s^2}{4\pi^3}z\left(\sin\frac{2\pi u}{s} - \sin\frac{2\pi(u+x)}{s} - \sin\frac{2\pi(u+y)}{s} - \sin\frac{2\pi(u+x+y)}{s}\right)\\ =&C_C^1(x,y,z;u,v,w) + C_C^2(x,y,z;u,v,w) + C_C^3(x,y,z;u,v,w) + C_C^4(x,y,z;u,v,w)\,,\end{aligned} \tag{2.24}$$

where

$$C_C^1(x,y,z;u,v,w) = \frac{s^2}{4\pi^3}\left(x\sin\frac{2\pi v}{s} + y\sin\frac{2\pi w}{s} + z\sin\frac{2\pi u}{s}\right)\,, \tag{2.25}$$

$$C_C^2(x,y,z;u,v,w) = -\frac{s^2}{4\pi^3}\left(x\sin\frac{2\pi(y+v)}{s} + y\sin\frac{2\pi(z+w)}{s} + z\sin\frac{2\pi(x+u)}{s}\right), \tag{2.26}$$

$$C_C^3(x,y,z;u,v,w) = -\frac{s^2}{4\pi^3}\left(x\sin\frac{2\pi(v+z)}{s} + y\sin\frac{2\pi(w+x)}{s} + z\sin\frac{2\pi(u+y)}{s}\right), \tag{2.27}$$

$$C_C^4(x,y,z;u,v,w) = \frac{s^2}{4\pi^3}\left(x\sin\frac{2\pi(y+v+z)}{s} + y\sin\frac{2\pi(z+w+x)}{s} + z\sin\frac{2\pi(x+u+y)}{s}\right), \tag{2.28}$$

and

$$s = x+y+z+u+v+w\,. \tag{2.29}$$

The valuable s is associated with the circumference of the semi-infinite cylinder.

According to Murata and Schnabl [12,13], the energy density of solution (2.15) is expected to be

$$\widehat{V}(\Psi) = \frac{1}{2\pi^2}\lim_{z\to 0} z\frac{G'(z)}{G(z)}, \tag{2.30}$$

$$G(z) = 1 - F^2(z)\,, \tag{2.31}$$

where $G(z)$ is supposed to be holomorphic for $\Re(z) \leq 0$ except for $z=0$. In reference [12], conditions on G or the application limit of this formula is discussed. Since we only consider the specific solution (1.3) in this paper, we refrain from going further into these general arguments.

3 Cutoff regularization

In what follows we concentrate on the following solution,

$$\Psi = \frac{1}{K}c\frac{K^2B}{K-1}c\,. \tag{3.1}$$

According to the formula (2.30), Ψ is expected to have an energy density corresponding to the double-brane background. In order to calculate the energy, we need to express the functions of K in (3.1) as a superposition of the wedge states. In this section, we introduce a cutoff Λ to express $1/K$,

$$F^2(K) = \frac{1}{K} \to \left(\frac{1}{K}\right)_\Lambda \equiv -\int_0^\Lambda dx\, e^{Kx}\,. \tag{3.2}$$

The other part, $H(K) = K^2/(K-1)$, is written as follows:

$$H(K) = \frac{K^2}{K-1} = -\int_0^\infty du\, e^{-u}\frac{\partial^2}{\partial u^2}e^{Ku}\,. \tag{3.3}$$

The regularized solution corresponding to (3.2) is given by

$$\Psi_{cutoff}(\Lambda) = \int_0^\Lambda dx\int_0^\infty du\, e^{-u}e^{xK}c\frac{\partial^2}{\partial u^2}e^{uK}Bc\,. \tag{3.4}$$

The regularized solution $\Psi_{cutoff} = \lim_{\Lambda\to\infty}\Psi_{cutoff}(\Lambda)$ do not satisfy the equation of motion, but calculation in this section will be useful later.

3.1 Kinetic term

Let us calculate the kinetic term for $\Psi_{cutoff}(\Lambda)$:

$$\begin{aligned}\widehat{V}_K(\Psi_{cutoff}(\Lambda),\ \Psi_{cutoff}(\Lambda')) &= \frac{\pi^2}{3}\,\mathrm{tr}\,[\,\Psi_{cutoff}(\Lambda)\,Q\,\Psi_{cutoff}(\Lambda')\,] \\ &= \frac{\pi^2}{3}\int_0^\infty du\int_0^\infty dv\,e^{-u-v}\int_0^\Lambda dx\int_0^{\Lambda'} dy\frac{\partial^2}{\partial u^2}\frac{\partial^2}{\partial v^2}C_K(x,\ y;\ u,\ v) \\ &= \frac{\pi^2}{3}\int_0^\infty du\int_0^\infty dv\,e^{-u-v}C_K^{(-1,-1,\,2,\,2)}(\Lambda,\ \Lambda';\ u,\ v)\,,\end{aligned} \tag{3.5}$$

where the braced upper suffices of the function denotes the order of the differentiation for each variable and the index -1 represents an integration (from zero to the variable); that is,

$$C_K^{(-1,-1,2,2)}(x\,,y;\ u,\ v) \equiv \int_0^x dx'\int_0^y dy'\,\frac{\partial^2}{\partial u^2}\frac{\partial^2}{\partial v^2}C_K(x',y';u,v)\,. \tag{3.6}$$

The explicit form of it is given by

$$\begin{aligned}&C_K^{(-1,-1,2,2)}(x,y;u,v) \\ =&C_K^{1\ (-1,-1,2,2)}(x,y;u,v)+C_K^{2\ (-1,-1,2,2)}(x,y;u,v)+C_K^{3\ (-1,-1,2,2)}(x,y;u,v)\,,\end{aligned} \tag{3.7}$$

where

$$\begin{aligned}&C_K^{i\ (-1,-1,2,\,2)}(x,\ y,\ u,\ v) \\ =&c_K^{i\ (-1,-1,2,\,2)}(x,\ y,\ u,\ v)-c_K^{i\ (-1,-1,2,\,2)}(0,\ y,\ u,\ v) \\ &-c_K^{i\ (-1,-1,2,\,2)}(x,\ 0,\ u,\ v)+c_K^{i\ (-1,-1,2,\,2)}(0,\ 0,\ u,\ v)\,,\end{aligned} \tag{3.8}$$

and

$$\begin{aligned}&c_K^{1\ (-1,-1,2,2)}(x,y;u,v) \\ =&\frac{1}{8\pi^6s^4}\Bigg\{2\pi x(2\pi^2yx^2+s^3)\cos\frac{2\pi x}{s}+s^2(2\pi^2x^2-s^2)\sin\frac{2\pi x}{s} \\ &+2\pi y(2\pi^2xy^2+s^3)\cos\frac{2\pi y}{s}+s^2(2\pi^2y^2-s^2)\sin\frac{2\pi y}{s}\Bigg\}\,,\end{aligned} \tag{3.9}$$

$$\begin{aligned}&c_K^{2\ (-1,-1,2,2)}(x,y;u,v) \\ =&\frac{1}{8\pi^6s^4}\Bigg\{-2\pi(s-u)(2\pi^2uv(s-u)+s^3)\cos\frac{2\pi u}{s} \\ &+s^2(2\pi^2(s-u)^2-s^2)\sin\frac{2\pi u}{s} \\ &-2\pi(s-v)(2\pi^2uv(s-v)+s^3)\cos\frac{2\pi v}{s} \\ &+s^2(2\pi^2(s-v)^2-s^2)\sin\frac{2\pi v}{s}\Bigg\}\,,\end{aligned} \tag{3.10}$$

$$c_K^{3\,(-1,-1,2,2)}(x,y;u,v) = \frac{1}{\pi^3 s^4}(x+u)(y+v)(xy-uv)\cos\frac{2\pi(x+u)}{s}\,. \tag{3.11}$$

It is straightforward to derive $\widehat{V}_K$ from $C_K{}^{(-1,-1,\,2,\,2)}(x,\,y;\,u,\,v)$. For large $\Lambda+\Lambda'$, $\widehat{V}_K$ is expressed as a function of Λ and Λ' as follows,

$$\begin{aligned}&\widehat{V}_K(\Psi_{cutoff}(\Lambda),\,\Psi_{cutoff}(\Lambda'))\\ \sim&\frac{1}{3}\left\{\left(1+\frac{2\Lambda\Lambda'\pi^2}{(\Lambda+\Lambda')^2}\right)\cos\left(\frac{2\Lambda\pi}{\Lambda+\Lambda'}\right)-\pi\left(1-\frac{2\Lambda}{\Lambda+\Lambda'}\right)\sin\left(\frac{2\Lambda\pi}{\Lambda+\Lambda'}\right)+2\right\},\end{aligned} \tag{3.12}$$

or putting $\Lambda'/\Lambda = a$,

$$\begin{aligned}&\widehat{\mathcal{V}}_K(a)\\ \equiv&\lim_{\Lambda\to\infty}\widehat{V}_K(\Psi_{cutoff}(\Lambda),\,\Psi_{cutoff}(a\Lambda))\\ =&\frac{1}{3}\left\{\left(1+\frac{2a\pi^2}{(1+a)^2}\right)\cos\left(\frac{2\pi}{1+a}\right)-\pi\left(1-\frac{2}{1+a}\right)\sin\left(\frac{2\pi}{1+a}\right)+2\right\}.\end{aligned} \tag{3.13}$$

The range of the value of $\widehat{\mathcal{V}}_K(a)$ is

$$\frac{2-\pi^2}{6} \le \widehat{\mathcal{V}}_K(a) < 1\,, \tag{3.14}$$

and for $a \to 0$ or ∞, the function converges to one,

$$\lim_{a\to\infty}\lim_{\Lambda\to\infty}\widehat{V}_K(\Psi_{cutoff}(\Lambda),\,\Psi_{cutoff}(a\Lambda)) = \lim_{a\to 0}\lim_{\Lambda\to\infty}\widehat{V}_K(\Psi_{cutoff}(\Lambda),\,\Psi_{cutoff}(a\Lambda)) = 1\,. \tag{3.15}$$

The minimum is realized when $a=1$,

$$\widehat{\mathcal{V}}_K(1) = \frac{2-\pi^2}{6} \quad (\sim -1.3116\ldots)\,, \tag{3.16}$$

which corresponds to equating the two cutoffs $\Lambda = \Lambda'$.

3.2 Cubic term

Let us move on to the cubic term $\widehat{V}_C$,

$$\begin{aligned}&\widehat{V}_C(\Psi_{cutoff}(\Lambda_1),\,\Psi_{cutoff}(\Lambda_2),\,\Psi_{cutoff}(\Lambda_3))\\ =&\int_0^{\Lambda_1}dx\int_0^{\Lambda_2}dy\int_0^{\Lambda_3}dz\int_0^{\infty}du\int_0^{\infty}dv\int_0^{\infty}dw\\ &\qquad\times e^{-u-v-w}\frac{\partial^2}{\partial u^2}\frac{\partial^2}{\partial v^2}\frac{\partial^2}{\partial w^2}C_C(x,y,z;u,v,w)\\ =&\int_0^{\infty}du\int_0^{\infty}dv\int_0^{\infty}dw e^{-u-v-w}C_C^{(-1,-1,-1,2,2,2)}(\Lambda_1,\,\Lambda_2,\,\Lambda_3;\,u,\,v,\,w)\,.\end{aligned} \tag{3.17}$$

We omit the intermediate calculations since it is basically parallel to that for the kinetic term. See appendix A for details. For large Λ_1, Λ_2, Λ_3, the cubic term $\widehat{V}_C$ is determined by a ratio of three cutoffs. Then we define the function $\mathcal{V}(a,b)$ by

$$\widehat{V}_C(\Psi_{cutoff}(\Lambda_1),\, \Psi_{cutoff}(\Lambda_2),\, \Psi_{cutoff}(\Lambda_3)) \equiv \widehat{\mathcal{V}}_C\left(\frac{\Lambda_2}{\Lambda_1},\, \frac{\Lambda_3}{\Lambda_1}\right) \qquad \text{(for large } \Lambda_i\text{'s)}\,, \tag{3.18}$$

and the explicit form of it is

$$\begin{aligned}
\widehat{\mathcal{V}}_C(a,\, b) \equiv & \lim_{\Lambda\to\infty} \widehat{V}_C(\Psi_{cutoff}(\Lambda),\, \Psi_{cutoff}(a\Lambda),\, \Psi_{cutoff}(b\Lambda)) \\
= & \; 1 + \frac{(1+a)^2+2a\pi^2}{3(1+a)^2}\cos\left(\frac{2\pi}{1+a}\right) + \frac{(1+b)^2+2b\pi^2}{3(1+b)^2}\cos\left(\frac{2\pi}{1+b}\right) \\
& + \frac{(a+b)^2+2ab\pi^2}{3(a+b)^2}\cos\left(\frac{2a\pi}{a+b}\right) \\
& + \frac{1}{3(1+a+b)^3}\Bigg(\{-(1+a+b)^2(-1+2a+2b)-2(a+b)^2\pi^2\}\cos\left(\frac{2\pi}{1+a+b}\right) \\
& \qquad + \{-(1+a+b)^2(2-a+2b)-2a(1+b)^2\pi^2\}\cos\left(\frac{2a\pi}{1+a+b}\right) \\
& \qquad + \{-(1+a+b)^2(2-b+2a)-2b(1+a)^2\pi^2\}\cos\left(\frac{2b\pi}{1+a+b}\right)\Bigg) \\
& + \frac{(1-a)\pi}{3(1+a)}\sin\left(\frac{2\pi}{1+a}\right) + \frac{(1-b)\pi}{3(1+b)}\sin\left(\frac{2\pi}{1+b}\right) \\
& + \frac{(a-b)\pi}{3(a+b)}\sin\left(\frac{2a\pi}{a+b}\right) \\
& + \frac{1}{6(1+a+b)^2\pi}\Bigg(\{3(1+a+b)^2-2(-2+a+b)(a+b)\pi^2\}\sin\left(\frac{2\pi}{1+a+b}\right) \\
& \qquad + \{3(1+a+b)^2-2(1+b-2a)(1+b)\pi^2\}\sin\left(\frac{2a\pi}{1+a+b}\right) \\
& \qquad + \{3(1+a+b)^2-2(1+a-2b)(1+a)\pi^2\}\sin\left(\frac{2b\pi}{1+a+b}\right)\Bigg)\,.
\end{aligned} \tag{3.19}$$

The range of the function $\widehat{\mathcal{V}}_C$ is

$$\frac{1}{2}-\frac{3\sqrt{3}}{4\pi}-\frac{19\pi^2}{54} \le \widehat{\mathcal{V}}_C(a,\, b) < 1\,. \tag{3.20}$$

The upper bound is realized as a limit when one of the three cutoffs is small enough compared to the other two:

$$\lim_{b\to 0}\left(\widehat{\mathcal{V}}_C(a,\, b)|_{a\neq 0}\right) = \lim_{a\to 0}\left(\widehat{\mathcal{V}}_C(a,\, b)|_{b\neq 0}\right) = \lim_{a,\, b\to\infty}\widehat{\mathcal{V}}_C(a,\, b) = 1\,, \tag{3.21}$$

and the minimum is realized when all the cutoffs are equated:

$$\widehat{\mathcal{V}}_C(1,\, 1) = \frac{1}{2}-\frac{3\sqrt{3}}{4\pi}-\frac{19\pi^2}{54} \qquad (\sim -3.38614)\,. \tag{3.22}$$

4 Regularization with general convergence factors

In this section, we consider a general class of regularizations and calculate the kinetic term $\hat{V}_K$ and the cubic term $\hat{V}_C$ for the regularized solution. We define the regularization R of the string field $1/K$ as follows:

$$K_R^{-1}(\Lambda) \equiv -\int_0^\infty R(\Lambda;\, x) e^{Kx} dx\,. \tag{4.1}$$

We impose the following three conditions on the convergence factor $R(\Lambda;\, x)$:

1. $\lim_{\Lambda\to\infty} R(\Lambda;\, x){=}1 \quad$ (for x fixed),
2. $\lim_{x\to\infty} R(\Lambda;\, x){=}0 \quad$ (for Λ fixed),
3. $R(\Lambda;\, x)$ is injective for $0 \le x \le \alpha(\Lambda)$ as a function of x, while $R(\Lambda;\, x)$ is zero for $\alpha(\Lambda) < x$, where $\alpha(\Lambda)$ is a positive number depending on Λ.

The third condition is somewhat constraining; it restrict the convergence factor $R(\Lambda; x)$ to be a monotonic decreasing function of x. (Note that the endpoint of the region $\alpha(\Lambda)$ is allowed to be infinity. In such a case, $R(\Lambda; x)$ must be injective for $0 \le x < \infty$. The typical example is $R(\Lambda; x) = \exp(-x/\Lambda)$.) We set

$$\Psi_R(\Lambda) \equiv K_R^{-1}(\Lambda) c \frac{K^2}{K-1} Bc\,, \tag{4.2}$$

and

$$\Psi_R \equiv \lim_{\Lambda\to\infty} \Psi_R(\Lambda)\,. \tag{4.3}$$

The following identity plays a central role in the discussion:

$$K_R^{-1}(\Lambda) = -\int_0^1 ds \int_0^{\lambda(\Lambda;\, s)} e^{Kx} dx = \int_0^1 ds \left(\frac{1}{K}\right)_{\lambda(\Lambda;\, s)}\,, \tag{4.4}$$

where the function $\lambda(\Lambda;\, s)$ is determined by the following equation:

$$\lambda(\Lambda;\, 1 - R(\Lambda;\, x)) = x\,. \tag{4.5}$$

Figure 1 schematically represents the relation between $R(\Lambda;\, x)$ and $\lambda(\Lambda;\, x)$. Then the formula (4.4) is derived as follows:

$$\begin{aligned}
-\int_0^1 ds \int_0^{\lambda(\Lambda;\, s)} e^{xK} dx &= \left[-s \int_0^{\lambda(\Lambda;\, s)} e^{xK} dx\right]_{s=0}^{s=1} + \int_0^1 dss \left(\frac{d}{ds}\int_0^{\lambda(\Lambda;\, s)} e^{xK} dx\right) \\
&= -\int_0^\alpha e^{xK} dx + \int_0^1 ds \frac{d\lambda(\Lambda;\, s)}{ds} s e^{\lambda(\Lambda;\, s)K} \\
&= -\int_0^\alpha R(\Lambda;\, s) e^{xK} dx\,.
\end{aligned}$$

[6] The discussion in Section 4 is inspired by the IMT variable transformation for singularity handling in the numerical integration by Iri, Moriguchi and Takasawa [19].

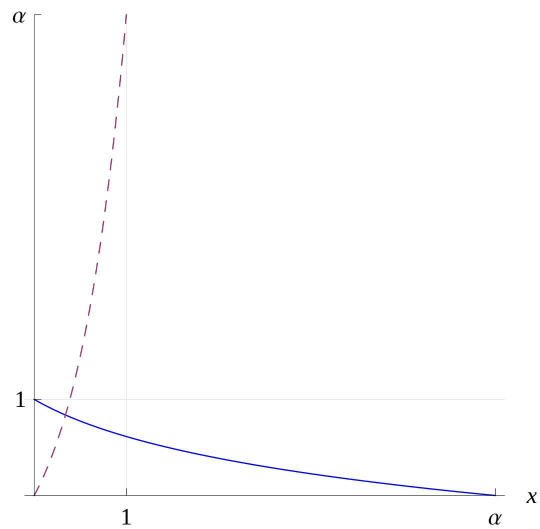

Figure 1: The solid line (blue) represents $R(\Lambda; x)$, while the dashed line (purple) represents $\lambda(\Lambda; x)$. These two curves are related by the $\pi/2$ rotation around the point (1/2, 1/2).

The relation (4.4) connects Ψ_R with Ψ_{cutoff} as follows:

$$\Psi_R(\Lambda) = \int_0^1 ds\, \Psi_{cutoff}(\lambda(\Lambda;\, s)). \tag{4.6}$$

Now, we can utilize the calculations in the previous section to evaluate the kinetic term $\hat{V}_K$ for Ψ_R,

$$\begin{aligned} \lim_{\Lambda\to\infty} \widehat{V}_K(\Psi_R(\Lambda),\, \Psi_R(\Lambda)) &= \lim_{\Lambda\to\infty} \int_0^1 ds_1 \int_0^1 ds_2\, \widehat{V}_K(\Psi_{cutoff}(\lambda(\Lambda;\, s_1),\, \Psi_{cutoff}(\lambda(\Lambda;\, s_2)) \\ &= \int_0^1 ds_1 \int_0^1 ds_2\, \widehat{\mathcal{V}}_K\left(\lim_{\Lambda\to\infty} \frac{\lambda(\Lambda;\, s_1)}{\lambda(\Lambda;\, s_2)}\right)\,, \end{aligned} \tag{4.7}$$

where the function $\widehat{\mathcal{V}}_K(a)$ is defined in (3.13). Then it can be said that the following function determines the value of the integral,

$$f(s_1,\, s_2) \equiv \lim_{\Lambda\to\infty} \frac{\lambda(\Lambda;\, s_1)}{\lambda(\Lambda;\, s_2)}\,. \tag{4.8}$$

From the expression (4.7) and (3.14), we obtain the following inequality:

$$\widehat{V}_K(\Psi_R) = \lim_{\Lambda\to\infty} \widehat{V}_K(\Psi_R(\Lambda),\, \Psi_R(\Lambda)) \le 1\,. \tag{4.9}$$

To gain the maximal value 1, the following condition is sufficient for $f(s_1,\, s_2)$,

$$f(s_1,\, s_2) = \begin{cases} \infty & s_1 > s_2\,, \\ 0 & s_2 > s_1\,. \end{cases} \tag{4.10}$$

and there exist some functions which satisfy the condition (4.10). For example, we take

$$\begin{aligned} R(\Lambda;\, x) &= \begin{cases} 1-\log(x+1)/\log(\Lambda+1) & (0\le x\le\Lambda) \\ 0 & (\Lambda < x) \end{cases} \\ &\equiv R_0(\Lambda;\, x)\,, \end{aligned} \tag{4.11}$$

and correspondingly,

$$\begin{aligned} \lambda(\Lambda;\, s) &= (\Lambda+1)^s - 1 \\ &\equiv \lambda_0(\Lambda;\, s)\,. \end{aligned} \tag{4.12}$$

Then, we obtain that

$$\lim_{\Lambda\to\infty} \widehat{V}_K(\Psi_{R_0}(\Lambda),\, \Psi_{R_0}(\Lambda)) = 1\,. \tag{4.13}$$

Note that, for arbitrary positive constant $a > 0$, the following identity holds:

$$\lim_{\Lambda\to\infty} \frac{\lambda_0(a\Lambda;\, s_1)}{\lambda_0(\Lambda;\, s_2)} = \lim_{\Lambda\to\infty} \frac{(a\Lambda+1)^{s_1}}{(\Lambda+1)^{s_2}} = \begin{cases} \infty & s_1 > s_2\,, \\ 0 & s_2 > s_1\,. \end{cases} \tag{4.14}$$

This expression tells us that the value of the kinetic term for Ψ_{R_0} does not depend on the ratio of two cutoff parameters:

$$\lim_{\Lambda\to\infty} \widehat{V}_K(\Psi_{R_0}(a\Lambda),\, \Psi_{R_0}(\Lambda)) = 1 \qquad \text{for } a > 0\,. \tag{4.15}$$

In the same way, we can derive the following relation for the cubic term:

$$\lim_{\Lambda\to\infty} \widehat{V}_C(\Psi_{R_0}(a\Lambda),\, \Psi_{R_0}(b\Lambda),\, \Psi_{R_0}(\Lambda)) = 1 \qquad \text{for } a,\, b > 0\,. \tag{4.16}$$

Therefore, we conclude that

$$\widehat{V}_K(\Psi_{R_0}) = \widehat{V}_C(\Psi_{R_0}) = 1\,. \tag{4.17}$$

In other words, Ψ_{R_0} satisfies the eom contracted with Ψ_{R_0} itself, and reproduces the expected energy for double D-branes.

5 Equation of motion contracted with states in the Fock space

So far we have seen that Ψ_{R_0} satisfies the equation of motion contracted with Ψ_{R_0} itself. Further investigation reveals that the eom contracted with some states in the Fock space are broken. To recover them, we need to add a correction term (a phantom piece). In Section 5.1 and Section 5.2, we calculate the remainder (or, an anomaly,) of the eom. In Section 5.3, we give an expression for the phantom piece φ_p and see that $\dot{\Psi} \equiv \Psi_{R_0} + \varphi_p$ satisfies the eom contracted with both solution itself and with the states in the Fock space.

5.1 Remainder of the equation of motion

In general, $\Psi_R(\Lambda)$ does not satisfy the eom for finite Λ. Let us define a remainder of the eom $\chi_{eom}(\Psi_R(\Lambda))$ as follows:

$$\chi_{eom}(\Psi_R(\Lambda)) \equiv Q\Psi_R(\Lambda) + \Psi_R(\Lambda) * \Psi_R(\Lambda)\,. \tag{5.1}$$

We investigate the contraction of $\chi_{eom}(\Psi_{R_0}(\Lambda))$ with a string field ϕ,

$$\begin{aligned} \mathrm{tr}[\,\chi_{eom}(\Psi_R(\Lambda))\,\phi\,] =& \mathrm{tr}[\,(Q\Psi_R(\Lambda) + \Psi_R(\Lambda) * \Psi_R(\Lambda))\,\phi\,] \\ =& \mathrm{tr}\left[\,K_R^{-1}(\Lambda)\,c\left(K + \frac{K^2}{K-1}K_R^{-1}(\Lambda) - \frac{K^2}{K-1}\right)c\frac{K^2}{K-1}Bc\,\phi\,\right] \\ &- \mathrm{tr}\left[\,K_R^{-1}(\Lambda)\,c\frac{K^2}{K-1}c\left(K + K_R^{-1}(\Lambda)\frac{K^2}{K-1} - \frac{K^2}{K-1}\right)Bc\,\phi\,\right]. \end{aligned} \tag{5.2}$$

For the string field Ψ_R to be a solution to the eom, the quantity $\mathrm{tr}[\,\chi_{eom}(\Psi_R(\Lambda))\,\phi\,]$ must vanish when we send Λ to infinity. In order to analyze $\mathrm{tr}[\,\chi_{eom}(\Psi_R(\Lambda))\,\phi\,]$, let us define the function $C_\phi(x,\,y)$ as follows:

$$C_\phi(x,\,y) \equiv \mathrm{tr}\left[\,e^{xK}ce^{yK}c\frac{K^2}{K-1}Bc\,\phi\,\right] - \mathrm{tr}\left[\,e^{xK}c\frac{K^2}{K-1}ce^{yK}Bc\,\phi\,\right]. \tag{5.3}$$

The right hand side of (5.2) is then given by

$$\begin{aligned} &\mathrm{tr}[\,\chi_{eom}(\Psi_R(\Lambda))\,\phi\,] \\ =& \int_0^\infty dx\,R(\Lambda;\,x) \\ &\times\left(\partial_y C_\phi(x,\,y)|_{y=0} + \int R(\Lambda;\,y)dy\int_0^\infty du e^{-u}\partial_u^2 C_\phi(x,\,y+u) + \int_0^\infty dy e^{-y}\partial_y^2 C_\phi(x,\,y)\right) \\ =& \int_0^\infty dx\int_0^\infty dy\,R(\Lambda,x)\,\frac{dR(\Lambda,y)}{dy}\int_0^\infty du e^{-u}\partial_u C_\phi(x,y+u)\,, \end{aligned} \tag{5.4}$$

where we used integration by parts,

$$\begin{aligned} &\int R(\Lambda;\,y)dy\int_0^\infty du e^{-u}\partial_u^2 C_\phi(x,y+u) \\ =& -\partial_{y'}C_\phi(x,y')|_{y'=0} - \int_0^\infty dy' e^{-y'}\partial_{y'}^2 C_\phi(x,y') \\ &- \int_0^\infty dy\frac{dR(\Lambda,y)}{dy}\partial_y C_\phi(x,y) - \int_0^\infty du\int_0^\infty dy\frac{dR(\Lambda,y)}{dy}e^{-u}\partial_u^2 C_\phi(x,y+u)\,. \end{aligned} \tag{5.5}$$

To go a step further, we use the following identities:

$$\int_0^\infty dx\,R(\Lambda;\,x)f(x) = \int_0^1 ds\int_0^{\lambda(\Lambda;\,s)} dx f(x)\,, \tag{5.6}$$

$$\int_0^\infty dx\,\frac{dR(\Lambda;\,x)}{dx}f(x) = -\int_0^1 ds f(\lambda(\Lambda;\,s))\,. \tag{5.7}$$

Then we obtain that

$$\mathrm{tr}[\,\chi_{eom}(\Psi_R(\Lambda))\,\phi\,] = -\int_0^1 ds_1 \int_0^1 ds_2 \int_0^{\lambda(\Lambda;\,s_1)} dx \int_0^\infty du e^{-u}\partial_u C_\phi(x, \lambda(\Lambda;\,s_2)+u)\,. \tag{5.8}$$

If we take the regularization R_0, then (5.8) is reduced to be

$$\mathrm{tr}[\,\chi_{eom}(\Psi_{R_0}(\Lambda))\,\phi\,] = -\frac{1}{2}\left(\lim_{a\to 0} + \lim_{a\to\infty}\right)\left(\lim_{\Lambda\to\infty}\int_0^\Lambda dx \int_0^\infty du e^{-u}\partial_u C_\phi(x,\,a\Lambda+u)\right). \tag{5.9}$$

5.2 The equation of motion contracted with states in the Fock space

Let us calculate the remainder of the eom for Ψ_{R_0}. For convenience, we use the $\mathcal{L}_0$ Fock basis instead of the L_0 Fock basis. It is obtained by acting finite number of generating operators written in $z = (2/\pi)\arctan\xi$ coordinates on the vacuum state $|0\rangle$. The $\mathcal{L}_0$ level of the state ϕ is defined by its $\mathcal{L}_0$ eigenvalue plus one. For example, the following state has $\mathcal{L}_0$ level 0:

$$\tilde{c}_1|0\rangle = \frac{\pi}{2}c_1|0\rangle\,. \tag{5.10}$$

As the first step, let us prove the following proposition: if the $\mathcal{L}_0$ level of the state ϕ is larger than one , then the remainder of the eom contracted with ϕ is zero. Suppose that the $\mathcal{L}_0$ level l of the state ϕ is larger than one. For large Λ, the function $\partial_u C_\phi(x,\,a\Lambda+u)$ can be evaluated as

$$\left|\partial_u C_\phi(x,\,a\Lambda+u)\right| < \frac{\alpha}{\Lambda^l}\,, \qquad \alpha:\ \mathrm{const.} \tag{5.11}$$

Then, we obtain

$$\lim_{\Lambda\to\infty}\int_0^\infty du\, e^{-u}\partial_u C_\phi(x,\,a\Lambda+u) = 0\,. \tag{5.12}$$

Since the limit is homogeneously convergent for $0 \le x \le x_0$, where x_0 is an arbitrary positive constant, we find that

$$\lim_{\Lambda\to\infty}\int_0^{x_0} dx \int_0^\infty du\, e^{-u}\partial_u C_\phi(x,\,a\Lambda+u) = 0\,. \tag{5.13}$$

On the other hand, for large x, we can suppress the integrand as

$$\left|\partial_u C_\phi(x, a\Lambda+u)\right| < \frac{\beta}{x^l}\,, \qquad \beta:\ \mathrm{const.} \tag{5.14}$$

Now let us divide the integration region of x into $[0, x_0]$ and $[x_0, \Lambda]$, and evaluate the integral. The result is

$$\begin{aligned}
&\left|\lim_{\Lambda\to\infty}\int_0^\Lambda dx \int_0^\infty du\, e^{-u}\partial_u C_\phi(x, a\Lambda+u)\right| \\
&\le \lim_{x_0\to\infty}\lim_{\Lambda\to\infty}\int_0^{x_0} dx \left|\int_0^\infty du e^{-u}\partial_u C_\phi(x, a\Lambda+u)\right| \\
&\quad + \lim_{x_0\to\infty}\lim_{\Lambda\to\infty}\int_{x_0}^{\Lambda} dx \left|\int_0^\infty du e^{-u}\partial_u C_\phi(x, a\Lambda+u)\right| \\
&\le 0 + \lim_{x_0\to\infty}\int_{x_0}^\infty dx \int_0^\infty du e^{-u}\frac{\beta}{x^l} \\
&=0\,.
\end{aligned} \tag{5.15}$$

Therefore, we conclude that $\mathrm{tr}[\,\chi_{eom}(\Psi_{R_0})\,\phi\,]=0$.

So we only need to calculate $\mathrm{tr}[\,\chi_{eom}(\Psi_{R_0})\,\phi\,]$ for states ϕ with $\mathcal{L}_0$ level lower than two. Let $\phi_{m,n}$ denote the states of the form

$$\phi_{n,m}=e^{K/2}K^n cK^m e^{K/2}. \tag{5.16}$$

After some calculation, we obtain

$$\begin{aligned}
\mathrm{tr}[\,\chi_{eom}(\Psi_{R_0})\,\phi_{0,\,0}\,] =& -\frac{1}{2}\left(\lim_{a\to 0}+\lim_{a\to\infty}\right)\lim_{\Lambda\to\infty}\int_0^\Lambda dx\int_0^\infty du\int_0^\infty dv\, e^{-u-v}\\
&\partial_u\partial_v^2\left\{F\left(\frac{1}{2},\,\frac{1}{2}+x,\,v,\,a\Lambda+u\right)-F\left(\frac{1}{2},\,\frac{1}{2}+x,\,a\Lambda+u,\,v\right)\right\}\\
=& -\frac{1}{2}\left(\lim_{a\to 0}+\lim_{a\to\infty}\right)\left(\frac{1}{2\pi^2}+\frac{1}{1+a}+\frac{-(1+a)^2+2\pi^2}{2(1+a)^2\pi^2}\cos\left(\frac{2a\pi}{1+a}\right)\right.\\
&+\frac{3+a}{2(1+a)\pi}\sin\left(\frac{2a\pi}{1+a}\right)+\mathrm{Ci}(2\pi)+\mathrm{Ci}\left(\frac{2a\pi}{1+a}\right)\\
&\left.+\log\left(\frac{1+a}{a}\right)\right)\\
=& -\frac{1}{2}(2-\gamma+\mathrm{Ci}(2\pi)-\log(2\pi)),
\end{aligned} \tag{5.17}$$

$$\begin{aligned}
\mathrm{tr}[\,\chi_{eom}(\Psi_{R_0})\,\phi_{1,\,0}\,] =& -\left(\lim_{a\to 0}+\lim_{a\to\infty}\right)\lim_{\Lambda\to\infty}\int_0^\Lambda dx\int_0^\infty du\int_0^\infty dv\, e^{-u-v}\\
&\partial_u\partial_v^2\left\{F^{(1,\,0,\,0,\,0)}\left(\frac{1}{2},\,\frac{1}{2}+x,\,v,\,a\Lambda+u\right)-F^{(1,\,0,\,0,\,0)}\left(\frac{1}{2},\,\frac{1}{2}+x,\,a\Lambda+u,\,v\right)\right\}\\
=& -\left(\lim_{a\to 0}+\lim_{a\to\infty}\right)\left(\frac{1}{2\pi^2}+\frac{1}{1+a}+\frac{-(1+a)^2+2\pi^2}{2(1+a)^2\pi^2}\cos\left(\frac{2a\pi}{1+a}\right)\right.\\
&+\frac{3+a}{2(1+a)\pi}\sin\left(\frac{2a\pi}{1+a}\right)+\mathrm{Ci}(2\pi)+\mathrm{Ci}\left(\frac{2a\pi}{1+a}\right)\\
&\left.+\log\left(\frac{1+a}{a}\right)\right)\\
=& -2+\gamma-\mathrm{Ci}(2\pi)+\log(2\pi)\,,
\end{aligned} \tag{5.18}$$

and

$$\mathrm{tr}[\,\chi(\Psi_{R_0})\,\phi_{0,\,1}\,]=0\,, \tag{5.19}$$

where $\mathrm{Ci}(x)$ is the cosine integral function

$$\mathrm{Ci}(x)=-\int_x^\infty\frac{\cos t}{t}dt\,,$$

and γ denotes the Euler-Mascheroni constant,

$$\gamma\equiv\lim_{n\to\infty}\left(\sum_{i=1}^n\frac{1}{i}-\ln n\right)=0.57721...$$

The correspondence between $\phi_{m,n}$ and the states in the $\mathcal{L}_0$ Fock space is given as follows:

$$\tilde{c}_1|0\rangle \sim e^{\frac{K}{2}} c e^{\frac{K}{2}} = \phi_{0,\,0}\,, \tag{5.20}$$

$$\mathcal{L}_{-1}\tilde{c}_1|0\rangle \sim e^{\frac{K}{2}}(Kc - cK)e^{\frac{K}{2}} = \phi_{1,0} - \phi_{0,1}\,. \tag{5.21}$$

We also see that the following quantity is zero,

$$\langle \chi_{eom}(\Psi_{R_0})|\, \mathcal{L}_{-1}^{\mathrm{matter}}\tilde{c}_1|0\rangle = 0\,, \tag{5.22}$$

since the matter one-point function vanishes. Then, we conclude that

$$\begin{aligned}
\langle \chi_{eom}(\Psi_{R_0})|\,\tilde{c}_1|0\rangle &= \frac{1}{2}(-2+\gamma-\mathrm{Ci}(2\pi)+\log(2\pi)) \qquad (\sim -0.218827)\,, && (5.23)\\
\langle \chi_{eom}(\Psi_{R_0})|\,\tilde{c}_0|0\rangle &= \frac{1}{2}(-2+\gamma-\mathrm{Ci}(2\pi)+\log(2\pi))\,, && (5.24)\\
\langle \chi_{eom}(\Psi_{R_0})|\,\phi_{\mathrm{other}}\rangle &= 0\,, && (5.25)
\end{aligned}$$

where ϕ_{other} denotes states belonging to the $\mathcal{L}_0$ Fock basis other than $\tilde{c}_1|0\rangle$ and $\tilde{c}_0|0\rangle$. The constants presented in these expressions can be gathered into a single series,

$$\gamma - \mathrm{Ci}(2\pi) + \log(2\pi) = -\frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k(2\pi)^{2k}}{k(2k)!}\,. \tag{5.26}$$

5.3 The phantom piece

In this subsection, we recover the violation of the equation of motion on the Fock space by adding some correction term, which is conventionally called a phantom piece. Since the remainder of the eom $\chi_{eom}(\Psi_{R_0})$ has only two components in $\mathcal{L}_0$ Fock basis for our regularization, we can find the phantom piece by brute-force analysis. Here we give an example of it, although it is not simple. We do not expect uniqueness of it; possibly there may exist other choices of the phantom piece, which can be more simple.

The explicit form of it is given by

$$\varphi_p(\Lambda) = r \int_0^\infty d\lambda \frac{dR_*(\Lambda;\, \lambda)}{d\lambda} \left(\frac{1}{K}\right)_\lambda cKe^{\lambda K} BcK\,, \tag{5.27}$$

where

$$R_*(\Lambda;\, \lambda) = R_1(\Lambda,\, \lambda) + \frac{i}{3^{\Lambda^2}}\, R_1(\Lambda,\, \tfrac{\lambda}{3^{2\Lambda^2}})\,, \tag{5.28}$$

$$R_1(\Lambda,\, x) = \begin{cases} 2 - (x+1)^{\frac{1}{\Lambda^2}} & (0 \le x \le 2^{\Lambda^2} - 1)\,, \\ 0 & (2^{\Lambda^2} - 1 < x)\,, \end{cases} \tag{5.29}$$

and

$$r = -\frac{2 - \gamma + \mathrm{Ci}(2\pi) - \ln(2\pi)}{2 - 4\mathrm{Ci}(\pi) + 4\mathrm{Ci}(2\pi) - 2\ln 4}\,. \tag{5.30}$$

As we see, φ_p possess an imaginary part for finite Λ. This is a subtle point, however, the imaginary part of components fields of φ_p vanishes in the limit $\Lambda \to \infty$. For our purpose, it is convenient to split φ_p into

$$\varphi_p = \varphi_p^{(1)} + \frac{i}{3^{\Lambda^2}}\, \varphi_p^{(2)}\,, \tag{5.31}$$

where

$$\varphi_p^{(i)} \equiv \lim_{\Lambda\to\infty} \varphi_p^{(i)}(\Lambda) \qquad i = 1,\, 2, \tag{5.32}$$

$$\varphi_p^{(1)}(\Lambda) = r \int_0^\infty d\lambda \frac{dR_1(\Lambda;\, \lambda)}{d\lambda} \left(\frac{1}{K}\right)_\lambda cKe^{\lambda K} BcK\,, \tag{5.33}$$

$$\varphi_p^{(2)}(\Lambda) = r \int_0^\infty d\lambda \frac{dR_1(\Lambda;\, \frac{\lambda}{3^{2\Lambda^2}})}{d\lambda} \left(\frac{1}{K}\right)_\lambda cKe^{\lambda K} BcK\,. \tag{5.34}$$

We can rewrite $\varphi_p^{(1)}$ and $\varphi_p^{(2)}$ as follows:

$$\varphi_p^{(1)}(\Lambda) = r \int_0^1 ds \left(\frac{1}{K}\right)_{\lambda_1(s;\,\Lambda)} cKe^{\lambda_1(s;\,\Lambda)K} BcK\,, \tag{5.35}$$

$$\varphi_p^{(2)}(\Lambda) = r \int_0^1 ds \left(\frac{1}{K}\right)_{\lambda_2(s;\,\Lambda)} cKe^{\lambda_2(s;\,\Lambda)K} BcK\,, \tag{5.36}$$

where

$$\lambda_1(s;\, \Lambda) = (1+s)^{\Lambda^2} - 1\,, \tag{5.37}$$

$$\lambda_2(s;\,\Lambda) = 3^{2\Lambda^2}\left((1+s)^{\Lambda^2} - 1\right)\,. \tag{5.38}$$

The functions λ_1 and λ_2 satisfy the same relation as (4.10); for $i = 1,\, 2$,

$$\lim_{\Lambda\to\infty}\frac{\lambda_i(\Lambda;\, s_1)}{\lambda_i(\Lambda;\, s_2)} = \begin{cases} 0 & (s_1 > s_2)\,, \\ \infty & (s_2 > 2_1)\,. \end{cases} \tag{5.39}$$

We also have

$$\lim_{\Lambda\to\infty}\frac{\lambda_i(\Lambda;\, s_1)}{\lambda_j(\Lambda;\, s_2)} = 0 \qquad i < j\,, \tag{5.40}$$

and

$$\lim_{\Lambda\to 0}\frac{\lambda_1(\Lambda;\, s)}{3^{2\Lambda^2}} = 0\,. \tag{5.41}$$

In the following, we sometimes omit the arguments Λ of $\varphi^{(i)}(\Lambda)$ and $\lambda_i(\Lambda;\, s)$ for simplicity.

The string field $\varphi_p^{(i)}$ do not couple to Ψ_{R_0} in the energy calculation; for $1 \le i,\, j,\, k \le 2$, we have

$$\widehat{V}_K(\Psi_{R_0},\, \varphi_p^{(i)}) = \widehat{V}_K(\varphi_p^{(i)},\, \varphi_p^{(j)}) = 0\,, \tag{5.42}$$

and

$$\widehat{V}_C(\Psi_{R_0},\, \Psi_{R_0},\, \varphi_p^{(i)}) = \widehat{V}_C(\Psi_{R_0},\, \varphi_p^{(i)},\, \varphi_p^{(j)}) = \widehat{V}_C(\varphi_p^{(i)},\, \varphi_p^{(j)},\, \varphi_p^{(k)}) = 0\,. \tag{5.43}$$

Hence, we see that $\dot{\Psi}$ satisfies the eom contracted with the solution itself, where

$$\dot{\Psi} = \Psi_R + \varphi_p\,. \tag{5.44}$$

The contribution of φ_p to the remainder of the eom on the Fock space exactly cancels that of Ψ_{R_0}. It is straightforward to derive following expressions:

$$\begin{aligned} \mathrm{tr}\left[(Q\varphi_p^{(1)})\phi_{0,\,0}\right] &= r\lim_{\Lambda\to\infty}\left(C_K^{(-1,\,1;\,1,\,0)}(\tfrac{1}{2}+\Lambda,\, \tfrac{1}{2};\, \Lambda,\, 0) - C_K^{(-1,\,1;\,1,\,0)}(\tfrac{1}{2},\, \tfrac{1}{2};\, \Lambda,\, 0)\right) \\ &= r\left(-\frac{1}{2} + \mathrm{Ci}(\pi) - \mathrm{Ci}(2\pi) + \ln(2)\right)\,, \end{aligned} \tag{5.45}$$

$$\begin{aligned} \mathrm{tr}\left[(Q\varphi_p^{(2)})\phi_{1,\,0}\right] &= r\lim_{\Lambda\to\infty}\left(C_K^{(-1,\,2;\,1,\,0)}(\tfrac{1}{2}+\Lambda,\, \tfrac{1}{2};\, \Lambda,\, 0) - C_K^{(-1,\,2;\,1,\,0)}(\tfrac{1}{2},\, \tfrac{1}{2};\, \Lambda,\, 0)\right) \\ &= r\left(-1 + 2\mathrm{Ci}(\pi) - 2\mathrm{Ci}(2\pi) + 2\ln(2)\right)\,, \end{aligned} \tag{5.46}$$

$$\begin{aligned} \mathrm{tr}\left[(Q\varphi_p^{(1)})\phi_{0,\,1}\right] &= r\lim_{\Lambda\to\infty}\left(C_K^{(0,\,1;\,1,\,0)}(\tfrac{1}{2}+\Lambda,\, \tfrac{1}{2};\, \Lambda,\, 0) - C_K^{(0,\,1;\,1,\,0)}(\tfrac{1}{2},\, \tfrac{1}{2};\, \Lambda,\, 0)\right) \\ &= 0\,. \end{aligned} \tag{5.47}$$

From (5.40), we can derive the following expressions:

$$\begin{aligned} \mathrm{tr}\left[(\varphi_p^{(1)}\varphi_p^{(2)})\,\phi_{0,\,0}\right] =& \lim_{\Lambda_2\to\infty}\lim_{\Lambda_1\to\infty} r\int_{\frac{1}{2}}^{\Lambda_1+\frac{1}{2}} dx \\ &\times\left(C_C^{(0,\,0,\,1;\,1,\,1,\,0)}(x,\, \Lambda_2,\, \tfrac{1}{2};\, \Lambda_1,\, \Lambda_2,\, 0) - C_C^{(0,\,0,\,1;\,1,\,1,\,0)}(x,\, 0,\, \tfrac{1}{2};\, \Lambda_1,\, \Lambda_2,\, 0)\right) \\ =&0\,, \end{aligned} \tag{5.48}$$

$$
\begin{aligned}
\operatorname{tr}\Big[\,(\varphi_p^{(i)}\Psi_R)\,\phi_{0,\,0}\,\Big] =& \lim_{\Lambda_2\to\infty}\lim_{\Lambda_1\to\infty} r\int_{\frac{1}{2}}^{\Lambda_1+\frac{1}{2}} dx\,\int_0^\infty du\, e^{-u}\\
&\times\Big(C_C^{(0,\,0,\,0;\,1,\,2,\,0)}(x,\,\Lambda_2,\,\tfrac{1}{2};\,\Lambda_1,\,v,\,0) - C_C^{(0,\,0,\,0;\,1,\,2,\,0)}(x,\,0,\,\tfrac{1}{2};\,\Lambda_1,\,v,\,0)\Big)\\
=&0\,,
\end{aligned}
\tag{5.49}
$$

$$
\begin{aligned}
\operatorname{tr}\Big[\,(\Psi_R\varphi^{(i)})\,\phi_{0,\,0}\,\Big] =& \lim_{\Lambda_2\to\infty}\lim_{\Lambda_1\to\infty}\int_{\frac{1}{2}}^{\Lambda_1+\frac{1}{2}} dx\,\int_0^{\Lambda_2} dy\,\int_0^\infty du\, e^{-u}\\
&\times C_C^{(0,\,0,\,1;\,2,\,1,\,0)}(x,\,y,\,\tfrac{1}{2};\,u,\,\Lambda_2,\,0)\\
=&0\,.
\end{aligned}
\tag{5.50}
$$

From (5.41), it follows that

$$
\lim_{\Lambda\to\infty}\frac{1}{3^{\Lambda^2}}\operatorname{tr}\Big[\,(\varphi^{(2)}\varphi^{(1)})\,\phi_{0,\,0}\,\Big] = 0\,.
\tag{5.51}
$$

We can also derive

$$
\lim_{\Lambda\to\infty}\bigg(\operatorname{tr}\Big[\,(\varphi^{(1)}\,\varphi^{(1)})\,\phi_{0,\,0}\,\Big] - \frac{1}{3^{2\Lambda^2}}\operatorname{tr}\Big[\,(\varphi^{(2)}\,\varphi^{(2)})\,\phi_{0,\,0}\,\Big]\bigg) = 0\,,
\tag{5.52}
$$

from the following relations:

$$
\begin{aligned}
&\operatorname{tr}\Big[\,(\varphi^{(1)}\varphi^{(1)})\,\phi_{0,\,0}\,\Big] - \int_0^\infty ds_1\,\int_0^\infty ds_2\, C_{ms}(\lambda_1(\Lambda;\,s_1),\,\lambda_1(\Lambda;\,s_2))\\
=&\lim_{\Lambda\to\infty}\int_0^\infty ds_1\,\int_0^\infty ds_2\,\int_0^{\lambda_1(\Lambda;\,s_1)} dx\,\int_0^{\lambda_1(\Lambda;\,s_2)} dy\\
&\times\bigg(C_C^{(0,\,1,\,0,\,1,\,1,\,0)}(x,\,\lambda_1(\Lambda;\,s_1),\,y,\,\lambda_1(\Lambda;\,s_2),\,\tfrac{1}{2},\,0) - C_{ms}(\lambda_1(\Lambda;\,s_1),\,\lambda_1(\Lambda;\,s_2))\bigg)\\
=&\frac{1}{2}\lim_{\Lambda_1\to\infty}\lim_{\Lambda_2\to\infty}\bigg(\int_0^{\Lambda_1} dx\,\int_0^{\Lambda_2} dy\, C_C^{(0,\,1,\,0,\,1,\,1,\,0)}(x,\,\Lambda_1,\,y,\,\Lambda_2,\,\tfrac{1}{2},\,0) - C_{ms}(\Lambda_1,\,\Lambda_2)\bigg)\\
&+\frac{1}{2}\lim_{\Lambda_2\to\infty}\lim_{\Lambda_1\to\infty}\bigg(\int_0^{\Lambda_1} dx\,\int_0^{\Lambda_2} dy\, C_C^{(0,\,1,\,0,\,1,\,1,\,0)}(x,\,\Lambda_1,\,y,\,\Lambda_2,\,\tfrac{1}{2},\,0) - C_{ms}(\Lambda_1,\,\Lambda_2)\bigg)\\
=&0\,,
\end{aligned}
\tag{5.53}
$$

$$
\begin{aligned}
&\mathrm{tr}\left[\,(\varphi^{(2)}\varphi^{(2)})\,\phi_{0,\,0}\right]-\int_0^\infty ds_1\int_0^\infty ds_2\,C_{ms}(\lambda_2(s_1),\,\lambda_2(s_2))\\
=&\int_0^\infty ds_1\int_0^\infty ds_2\\
&\left(\int_0^{\lambda_2(s_1)}dx\int_0^{\lambda_2(s_2)}dy\,C_C^{(0,\,1,\,0,\,1,\,1,\,0)}(x,\,\lambda_2(s_1),\,y,\,\lambda_2(s_2),\,\tfrac{1}{2},\,0)-C_{ms}(\lambda_2(s_1),\,\lambda_2(s_2))\right)\\
=&\frac{1}{2}\lim_{\Lambda_1\to\infty}\lim_{\Lambda_2\to\infty}\left(\int_0^{\Lambda_1}dx\int_0^{\Lambda_2}dy\,C_C^{(0,\,1,\,0,\,1,\,1,\,0)}(x,\,\Lambda_1,\,y,\,\Lambda_2,\,\tfrac{1}{2},\,0)-C_{ms}(\Lambda_1,\,\Lambda_2)\right)\\
&+\frac{1}{2}\lim_{\Lambda_2\to\infty}\lim_{\Lambda_1\to\infty}\left(\int_0^{\Lambda_1}dx\int_0^{\Lambda_2}dy\,C_C^{(0,\,1,\,0,\,1,\,1,\,0)}(x,\,\Lambda_1,\,y,\,\Lambda_2,\,\tfrac{1}{2},\,0)-C_{ms}(\Lambda_1,\,\Lambda_2)\right)\\
=&0\,,
\end{aligned}
\tag{5.54}
$$

$$
C_{ms}(\lambda_1(s_1),\,\lambda_1(s_2))=\frac{1}{3^{2\Lambda^2}}C_{ms}(\lambda_2(s_1),\,\lambda_2(s_2))\,,
\tag{5.55}
$$

where

$$
\begin{aligned}
C_{ms}(x,\,y)=&\frac{x}{2\pi}+\frac{x+y}{\pi^2}\cos\left(\frac{\pi x}{x+y}\right)+\frac{x-2y}{2\pi^2}\cos\left(\frac{2\pi x}{x+y}\right)\\
&-\frac{x+y}{\pi}\sin\left(\frac{\pi x}{x+y}\right)-\frac{(x+y)^2+\pi^2(x^2+4xy+2y^2)}{\pi^3(x+y)}\sin\left(\frac{2\pi x}{x+y}\right)\\
&+\frac{x+2y}{\pi}\sin\left(\frac{\pi x}{x+2y}\right)+\frac{x+2y}{\pi}\sin\left(\frac{2\pi x}{x+2y}\right)\\
&-2y\Bigg(\mathrm{Ci}(2\pi)-\mathrm{Ci}\left(\frac{\pi y}{x+y}\right)+2\mathrm{Ci}\left(\frac{2\pi y}{x+y}\right)+\mathrm{Ci}\left(\frac{2\pi y}{x+2y}\right)\\
&-2\mathrm{Ci}\left(\frac{4\pi y}{x+2y}\right)-\mathrm{Ci}\left(\frac{\pi(x+2y)}{x+y}\right)\Bigg)\,.
\end{aligned}
\tag{5.56}
$$

From (5.45)-(5.52), we obtain that

$$
\mathrm{tr}[\,\chi_{eom}(\dot{\Psi})\,\phi_{0,\,0}\,]=0\,,
\tag{5.57}
$$

$$
\mathrm{tr}[\,\chi_{eom}(\dot{\Psi})\,\phi_{1,\,0}\,]=0\,,
\tag{5.58}
$$

$$
\mathrm{tr}[\,\chi_{eom}(\dot{\Psi})\,\phi_{0,\,1}\,]=0\,.
\tag{5.59}
$$

In addition, using the scaling property, we can also derive that

$$
\mathrm{tr}[\,\chi_{eom}(\dot{\Psi})\,\phi_{other}\,]=0\,.
\tag{5.60}
$$

Therefore, we conclude that $\dot{\Psi}$ satisfies the eom contracted to the solution itself and that contracted to the states in the Fock space. Note that our phantom piece is designed for the non-real solution Ψ_{R_0}. If we care about the reality condition, the form of the phantom piece may be subject to modification.

6 Remarks on component fields and Ellwood invariant

6.1 Component fields

It is curious that the component fields of Ψ_R do not depend on the choice of the convergence factor R. This fact can easily be shown as below. Let ϕ be a ghost-number-two state in the Fock space. Then,

$$\begin{aligned} \mathrm{tr}[\,\Psi_R,\,\phi\,] &= -\lim_{\Lambda\to\infty}\int_0^\infty dx\,R(\Lambda,\,x)\mathrm{tr}[\,e^{xK}c\frac{K^2}{K-1}Bc,\,\phi\,] \\ &= -\int_0^\infty dx\,\mathrm{tr}[\,e^{xK}c\frac{K^2}{K-1}Bc,\,\phi\,]\,. \end{aligned} \tag{6.1}$$

The integral in the last line absolutely converge for any choice of ϕ, and it does not depend on R. The tachyon component of Ψ_R is

$$\frac{\pi}{2}\mathrm{tr}[\,\Psi_R\,e^{\frac{K}{2}}cKce^{\frac{K}{2}}\,] \sim -0.37299\,. \tag{6.2}$$

This number is already obtained, though it is an intermediate in their calculation, in [12] by Murata and Schnabl (the relative minus sign is due to the notational difference).

Contrary to its name, some components of φ_p are not zero. Indeed, the tachyon component of φ_p is given by

$$\frac{\pi}{2}\mathrm{tr}[\,\varphi_p\,e^{\frac{K}{2}}cKce^{\frac{K}{2}}\,] \sim +0.171867\,. \tag{6.3}$$

6.2 Ellwood invariant

There is one more thing that is important. As explained below, the Ellwood invariant [7] [20] for the regularized solution $\dot{\Psi}$ is the same as that for pure-gauge solutions.

The Ellwood invariant is defined as follows:

$$\mathcal{W}(\Psi,\,\phi_{closed}) = \langle \mathcal{I}|\phi_{closed}(i)|\Psi\rangle\,, \tag{6.4}$$

where ϕ_{closed} represents an on-shell closed-string vertex operator. In particular, we simply define the Ellwood invariant for a regularized solution $\lim_{\Lambda\to\infty}\Psi(\Lambda)$ as

$$\mathcal{W}(\Psi,\,\phi_{closed}) = \lim_{\Lambda\to\infty}\langle \mathcal{I}|\phi_{closed}(i)|\Psi(\Lambda)\rangle\,. \tag{6.5}$$

In [20], Ellwood argued that the invariant (6.4) represents the difference of the closed string tadpole amplitude evaluated in the background corresponding to Ψ, and that evaluated in the perturbative vacuum:

$$\mathcal{W}(\Psi,\,\phi_{closed}) = \mathcal{A}_\Psi^{\mathrm{disk}}(\phi_{closed}) - \mathcal{A}_0^{disk}(\phi_{closed})\,. \tag{6.6}$$

[7] The gauge invariant observable, which is here called an Ellwood invariant, is independently found by Hashimoto and Itzhaki [21] and by Gaiotto, Rastelli, Sen and Zwiebach [22]. Afterward, its physical interpretation was given by Ellwood [20].

Then, for our solution $\dot{\Psi}$ we expect that

$$\begin{aligned}\mathcal{W}(\dot{\Psi},\,\phi_{closed}) =& 2\mathcal{A}_0^{\mathrm{disk}}(\phi_{closed}) - \mathcal{A}_0^{\mathrm{disk}}(\phi_{closed}) \\ =& \mathcal{A}_0^{\mathrm{disk}}(\phi_{closed}) \qquad \text{(expectation)}\,.\end{aligned} \tag{6.7}$$

However, this is not the case. Remember that the Ellwood invariants for solutions of the form

$$\Psi = \sum_i F_i(K)^2 c H_i(K) Bc\,, \tag{6.8}$$

are given by

$$\mathcal{W}(\Psi,\,\phi_{closed}) = \lim_{x\to 0} \sum_i \frac{d(F_i(x)^2)}{dx} H_i(x) \mathcal{A}_0^{\mathrm{disk}}(\phi_{closed})\,. \tag{6.9}$$

For derivation of this formula, see [12,13]. Note that we do not need the eom during the derivation of (6.9). If the solution possess a regularization parameter Λ, then the formula (6.9) becomes as follows:

$$\mathcal{W}(\Psi,\,\phi_{closed}) = \lim_{\Lambda\to\infty} \lim_{x\to 0} \sum_i \frac{d(F_i(x;\,\Lambda)^2)}{dx} H_i(x;\,\Lambda) \mathcal{A}_0^{\mathrm{disk}}(\phi_{closed})\,. \tag{6.10}$$

For the regularized double-brane solution $\dot{\Psi}$, the corresponding functions $H_i(x;\,\Lambda)$ have a zero at $x=0$ and $F_i^2(x;\,\Lambda)$ have no singularity at $x=0$.[8] Therefore, the Ellwood invariant for $\dot{\Psi}$ is zero,

$$\mathcal{W}(\dot{\Psi},\,\phi_{closed}) = 0\,. \tag{6.11}$$

Finally, we note that if we freely switch the order of limits, then we can obtain the desired result. That is,

$$\lim_{x\to 0}\left(\lim_{\Lambda\to\infty} \sum_i \frac{d(F_i(x;\,\Lambda)^2)}{dx} H_i(x;\,\Lambda) \mathcal{A}_0^{\mathrm{disk}}(\phi_{closed})\right) = \mathcal{A}_0^{\mathrm{disk}}(\phi_{closed})\,, \tag{6.12}$$

which coincides with (6.7).

7 Conclusion

Up to this work, the anomaly of the equation of motion was considered as a serious obstruction to construct the multiple-brane solutions. We explicitly regularize the double-brane solution and show that the regularized solution satisfies the eom when it is contracted to the solution itself and the states in the Fock space. We expect that this result can be generalized to the multiple-brane solutions with multiplicity more than two.

This paper argued that $\dot{\Psi}$ respects the eom, while the Ellwood invariant corresponding to the double-brane background is not reproduced. These results do not accord. We have to understand the reason. One possibility is to assume that the equation of motion is broken in some unspecified manner.

[8] Here we are assuming that the formula (6.9) also holds when the summation symbols on the right hand side of (6.8) and (6.9) is replaced by integrals.

Another possibility is that the Ellwood correspondence is subject to some change. It is worth stating that, according to Takahashi [23] or our recent paper [24], this mismatching is a typical example of the general results for wedge based solutions to the eom in KBc subalgebra.

There appeared recently two interesting papers on related subjects. In [25], Baba and Ishibashi discussed the relation between energy and the Ellwood invariant, and proved the Ellwood correspondence in part. Then their work might be key to resolve the above-mentioned paradox on the Ellwood invariant.

In [26], Hata and Kojita proposed a new energy formula for the Okawa-type solutions based on some novel symmetry of correlation functions, which also can be regarded as a symmetry of the Okawa-type solutions. According to their discussion, the following solution possess energy for double D-branes, since it is a symmetrical counterpart of the solution (1.3):

$$\Psi = Kc\frac{KB}{1-K}c\,. \tag{7.13}$$

We are now investigating this solution in detail, and actually it seems to possess energy for double D-branes. More over, this solution is even preferable in a sense, since it does not need any cumbersome phantom piece. In the forthcoming paper [27], we will report these results.

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A Cubic term

In this appendix, we present intermediate calculation omitted in Section 3.2. The cubic term $\widehat{V}_C$ for $\Psi_{cutoff}(\Lambda)$ is give by

$$
\begin{aligned}
&\widehat{V}_C(\Psi_{cutoff}(\Lambda_1),\, \Psi_{cutoff}(\Lambda_2),\, \Psi_{cutoff}(\Lambda_3)) \\
=&\int_0^{\Lambda_1} dx \int_0^{\Lambda_2} dy \int_0^{\Lambda_3} dz \int_0^{\infty} du \int_0^{\infty} dv \int_0^{\infty} dw \\
&\qquad \times e^{-u-v-w} \frac{\partial^2}{\partial u^2}\frac{\partial^2}{\partial v^2}\frac{\partial^2}{\partial w^2} C_C(x,y,z;u,v,w) \\
=&\int_0^{\infty} du \int_0^{\infty} dv \int_0^{\infty} dw e^{-u-v-w} C_C^{(-1,-1,-1,2,2,2)}(\Lambda_1,\, \Lambda_2,\, \Lambda_3;\, u,\, v,\, w)\,.
\end{aligned}
\tag{A.14}
$$

The explicit form of the function $C_C^{(-1,-1,-1,\,2,\,2,\,2)}(x,\, y,\, z;\, u,\, v,\, w)$ is given as follows:

$$
\begin{aligned}
&C_C^{(-1,-1,-1,2,2,2)}(x,y,z;u,v,w) \\
=&C^1{}_C^{(-1,-1,-1,2,2,2)}(x,y,z;u,v,w) + C^2{}_C^{(-1,-1,-1,2,2,2)}(x,y,z;u,v,w) \\
&+ C^3{}_C^{(-1,-1,-1,2,2,2)}(x,y,z;u,v,w) + C^4{}_C^{(-1,-1,-1,2,2,2)}(x,y,z;u,v,w)\,,
\end{aligned}
\tag{A.15}
$$

where

$$
\begin{aligned}
&C^i{}_C^{(-1,-1,-1,2,2,2)}(x,y,z;u,v,w) \\
=&c^i{}_C^{(-1,-1,-1,2,2,2)}(x,y,z;u,v,w) \\
&- c^i{}_C^{(-1,-1,-1,2,2,2)}(0,y,z;u,v,w) - c^i{}_C^{(-1,-1,-1,2,2,2)}(x,0,z;u,v,w) \\
&- c^i{}_C^{(-1,-1,-1,2,2,2)}(x,y,0;u,v,w) \\
&+ c^i{}_C^{(-1,-1,-1,2,2,2)}(0,0,z;u,v,w) + c^i{}_C^{(-1,-1,-1,2,2,2)}(x,0,0;u,v,w) \\
&+ c^i{}_C^{(-1,-1,-1,2,2,2)}(0,y,0;u,v,w) \\
&- c^i{}_C^{(-1,-1,-1,2,2,2)}(0,0,0;u,v,w)\,,
\end{aligned}
\tag{A.16}
$$

and

$$
\begin{aligned}
&c^1{}_C^{(-1,-1,-1,2,2,2)}(x,y,z;u,v,w) \\
&= \frac{1}{2\pi^3 s^4}\Bigg(2\pi(s-w)\{-w^3 - 3w^2(s-w) - (s-w)^3 - 3w(s-w)^2 + 2\pi^2 xw(s-w)\}\cos\frac{2\pi w}{s} \\
&\quad - s^2\{w^2 + 2w(s-w) - (2\pi^2-1)(s-w)^2\}\sin\frac{2\pi w}{s} \\
&\quad + 2\pi(s-u)\{-u^3 - 3u^2(s-u) - (s-u)^3 - 3u(s-u)^2 + 2\pi^2 yu(s-u)\}\cos\frac{2\pi u}{s} \\
&\quad - s^2\{u^2 + 2u(s-u) - (2\pi^2-1)(s-u)^2\}\sin\frac{2\pi u}{s} \\
&\quad + 2\pi(s-v)\{-v^3 - 3v^2(s-v) - (s-v)^3 - 3v(s-v)^2 + 2\pi^2 zv(s-v)\}\cos\frac{2\pi v}{s} \\
&\quad - s^2\{v^2 + 2v(s-v) - (2\pi^2-1)(s-v)^2\}\sin\frac{2\pi v}{s} \Bigg)\,,
\end{aligned}
\tag{A.17}
$$

$$
\begin{aligned}
&c^{2}{}_{C}^{(-1,-1,-1,2,2,2)}(x,y,z;u,v,w)\\
&= -\frac{1}{2\pi^3 s^4}\Bigg(\pi\{4\pi^2 x(s-w-z)(w+z)^2+s^3(s-2(w+z))\}\cos\frac{2\pi(w+z)}{s}\\
&\quad + s^2\{s^2-2\pi^2(s-w-z)(w+z)\}\sin\frac{2\pi(w+z)}{s}\\
&\quad + \pi\{4\pi^2 y(s-u-x)(u+x)^2+s^3(s-2(u+x))\}\cos\frac{2\pi(u+x)}{s}\\
&\quad + s^2\{s^2-2\pi^2(s-u-x)(u+x)\}\sin\frac{2\pi(u+x)}{s}\\
&\quad + \pi\{4\pi^2 z(s-v-y)(v+y)^2+s^3(s-2(v+y))\}\cos\frac{2\pi(v+y)}{s}\\
&\quad + s^2\{s^2-2\pi^2(s-v-y)(v+y)\}\sin\frac{2\pi(v+y)}{s}\Bigg),
\end{aligned}
\tag{A.18}
$$

$$
\begin{aligned}
&c^{3}{}_{C}^{(-1,-1,-1,2,2,2)}(x,y,z;u,v,w)\\
&= -\frac{1}{2\pi^3 s^4}\Bigg(\pi\{4\pi^2 x(s-w-y)(w+y)^2+s^3(s-2(w+y))\}\cos\frac{2\pi(w+y)}{s}\\
&\quad + s^2\{s^2+2\pi^2(s-w-y)(w+y)\}\sin\frac{2\pi(w+y)}{s}\\
&\quad + \pi\{4\pi^2 y(s-u-z)(u+z)^2+s^3(s-2(u+z))\}\cos\frac{2\pi(u+z)}{s}\\
&\quad + s^2\{s^2+2\pi^2(s-u-z)(u+z)\}\sin\frac{2\pi(u+z)}{s}\\
&\quad + \pi\{4\pi^2 x(s-v-x)(v+x)^2+s^3(s-2(v+x))\}\cos\frac{2\pi(v+x)}{s}\\
&\quad + s^2\{s^2+2\pi^2(s-v-x)(v+x)\}\sin\frac{2\pi(v+x)}{s}\Bigg),
\end{aligned}
\tag{A.19}
$$

$$
\begin{aligned}
&c^{4}{}_{C}^{(-1,-1,-1,2,2,2)}(x,y,z;u,v,w)\\
&= \frac{1}{2\pi^3 s^4}\Bigg(-2\pi(w+y+z)\{s^3+2\pi^2 x(w+y+z)^2)\}\cos\frac{2\pi(w+y+z)}{s}\\
&\quad + s^2\{s^2-2\pi^2(w+y+z)^2\}\sin\frac{2\pi(w+y+z)}{s}\\
&\quad - 2\pi(u+z+x)\{s^3+2\pi^2 y(u+z+x)^2)\}\cos\frac{2\pi(u+z+x)}{s}\\
&\quad + s^2\{s^2-2\pi^2(u+z+x)^2\}\sin\frac{2\pi(u+z+x)}{s}\\
&\quad - 2\pi(v+x+y)\{s^3+2\pi^2 x(v+x+y)^2)\}\cos\frac{2\pi(v+x+y)}{s}\\
&\quad + s^2\{s^2-2\pi^2(v+x+y)^2\}\sin\frac{2\pi(v+x+y)}{s}\Bigg).
\end{aligned}
\tag{A.20}
$$

For large Λ_1, Λ_2, Λ_3, $\widehat{V}_C$ becomes as follows:

$$
\begin{aligned}
&\lim_{\Lambda\to\infty}\widehat{V}_C(\Psi_{cutoff}(a\Lambda),\,\Psi_{cutoff}(b\Lambda),\,\Psi_{cutoff}(c\Lambda))\\
=&1+\frac{(a+b)^2+2ab\pi^2}{3(a+b)^2}\cos\left(\frac{2a\pi}{a+b}\right)+\frac{(b+c)^2+2bc\pi^2}{3(b+c)^2}\cos\left(\frac{2b\pi}{b+c}\right)\\
&+\frac{(c+a)^2+2ca\pi^2}{3(c+a)^2}\cos\left(\frac{2c\pi}{c+a}\right)\\
&+\frac{1}{3(a+b+c)^3}\Bigg(\{-(a+b+c)^2(-a+2b+2c)-2a(b+c)^2\pi^2\}\cos\left(\frac{2a\pi}{a+b+c}\right)\\
&\qquad+\{-(a+b+c)^2(-b+2c+2a)-2b(c+a)^2\pi^2\}\cos\left(\frac{2b\pi}{a+b+c}\right)\\
&\qquad+\{-(a+b+c)^2(-c+2a+2b)-2c(a+b)^2\pi^2\}\cos\left(\frac{2c\pi}{a+b+c}\right)\Bigg)\\
&+\frac{(a-b)\pi}{3(a+b)}\sin\left(\frac{2a\pi}{a+b}\right)+\frac{(b-c)\pi}{3(b+c)}\sin\left(\frac{2b\pi}{b+c}\right)\\
&+\frac{(c-a)\pi}{3(c+a)}\sin\left(\frac{2c\pi}{c+a}\right)\\
&+\frac{1}{6(a+b+c)^2\pi}\Bigg(\{3(a+b+c)^2-2(-2a+b+c)(b+c)\pi^2\}\sin\left(\frac{2a\pi}{a+b+c}\right)\\
&\qquad+\{3(a+b+c)^2-2(-2b+c+a)(c+a)\pi^2\}\sin\left(\frac{2b\pi}{a+b+c}\right)\\
&\qquad+\{3(a+b+c)^2-2(-2c+a+b)(a+b)\pi^2\}\sin\left(\frac{2c\pi}{a+b+c}\right)\Bigg).
\end{aligned}
\tag{A.21}
$$

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